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Interpolation of q-Variate Homogeneous Random Fields

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1. INTRODUCTION

Let $\mathscr{H}$ be a Hilbert space over the field of complex numbers, $\mathscr{H}^q$ $(1 \leqslant q < \infty)$ be the Cartesian product of $\mathscr{H}$ with itself q times. If $\mathbf{X}$, $\mathbf{Y}$ are in $\mathscr{H}^q$ and $\mathbf{A}$, $\mathbf{B}$ are $q \times q$ matrices with complex entries then $\mathbf{AX} + \mathbf{BY}$ is in $\mathscr{H}^q$. For $\mathbf{X}$, $\mathbf{Y}$ in $\mathscr{H}^q$, we define the Gramian of $\mathbf{X}$ and $\mathbf{Y}$ by

$$(\mathbf{X}, \mathbf{Y}) = [(x_i, y_j)], \qquad 1 \leqslant i, \qquad j \leqslant q,$$

where (x_i, y_j) is the inner product of x_i, y_j in $\mathscr{H}$. In $\mathscr{H}^q$ we define the inner product and norm by

$$((\mathbf{X}, \mathbf{Y})) = \operatorname{tr}(\mathbf{X}, \mathbf{Y}), \qquad \| \mathbf{X} \| = \operatorname{tr}(\mathbf{X}, \mathbf{X}), \qquad \operatorname{tr} = \text{trace}.$$

It is known that with this inner product $\mathscr{H}^q$ becomes a Hilbert space {cf. [1, Section 5]}.

Let $\mathbf{X}_k$ be an element of $\mathscr{H}^q$ for each lattice point k in Z^m {Z^m denotes the set of all latice points in the m-dimensional Euclidean space R^m}. We say that $\mathbf{X}_k$ is a q-variate homogeneous random field over Z^m if for all lattice points m, n and k we have

$$(\mathbf{X}_{m+k}, \mathbf{X}_{n+k}) = (\mathbf{X}_m, \mathbf{X}_n). \tag{1}$$

In this case we define

$$\mathbf{R}_k = (\mathbf{X}_k, \mathbf{X}_0). \tag{2}$$

It is known {cf. [2, Section 2]} that $\mathbf{R}_k$ can be written in the form

$$\mathbf{R}_k = \frac{1}{(2\pi)^m} \underbrace{\int^{2\pi} \cdots \int_0^{2\pi}}_{m} e^{-i(k,x)} \, d\mathbf{F}(x), \tag{3}$$

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where $(k, x) = \sum_{i=1}^{m} k_i x_i$ and, $d\mathbf{F}(x)$ is a nonnegative hermitian $q \times q$ matrix-valued measure defined on the Borel subsets of the m-dimensional torus

$$T_m = \{x : x = (x_i)_{i=1}^{m}, 0 < x_i \leqslant 2\pi \text{ and } 1 \leqslant i \leqslant m\}.$$

The following prediction theorem has been solved by H. Helson and D. Lowdenslager {cf. [3, Section 2]}.

1.1. THEOREM. *Let S be a half plane of lattice points in R^2 and let μ be a finite nonnegative measure on the 2-dimensional torus T_2. Let μ have Lebesgue decomposition*

$$d\mu = \omega(e^{ix}, e^{iy})\, d\sigma + d\mu_s(x, y),$$

where ω is nonnegative and summable for the measure $d\sigma = dx\, dy/4\pi^2$, and μ_s is singular with respect to $d\sigma$. Then

$$\exp\left\{\int_{T_2} \log \omega\, d\sigma\right\} = \inf_P \int_{T_2} |1 - P|^2\, d\mu, \tag{4}$$

where P ranges over finite sums of the form

$$P(e^{ix}, e^{iy}) = \sum_S a_{mn} e^{-i(mx+ny)}.$$

Now if $\mathbf{X}_{mn}$ is a univariate homogeneous random field over Z^2 with the correlation function $\mathbf{R}_{mn}$ and spectral distribution function $\mathbf{F}$ defined on the 2-dimensional torus T_2 it easily follows from Theorem 1.1 (4) that

$$\exp\left\{\int_{T_2} \log \omega\, d\sigma\right\} = \inf_{(m,n)\in S} \left\| \mathbf{X}_{00} - \sum a_{mn}\mathbf{X}_{mn} \right\|, \tag{5}$$

where $\sum a_{mn}\mathbf{X}_{mn}$ is a finite summand and S, as before, is a half-plane.

In this paper we propose to obtain results similar to (5) for the case that $\mathbf{X}_k$ is a q-variate $(1 \leqslant q < \infty)$ homogeneous random field over Z^m, $(1 \leqslant m < \infty)$ and S is a subset of Z^m whose complement in Z^m is bounded. These generalize the results given in [4, Section 10] and [5] concerning the interpolation of a q-variate stationary stochastic process and in particular when the complement of S consists of only a single point it gives rise to an extension of Kolmogorov's result {cf. [6, Theorem 24]}, Masani's result {cf. [7, Section 2]} on minimality of a stationary stochastic process.

2. Preliminaries

Let $\{\mathbf{X}_k, k \in Z^m\}$ be an $\mathscr{H}^q$-valued homogeneous random field over Z^m. We will denote its correlation matrix-valued function by $\mathbf{R}$ which is defined on Z^m. Its spectral distribution $q \times q$ matrix-valued function $\mathbf{F}$ is defined on the m-dimensional torus T_m. $\mathbf{F}$ induces a unique nonnegative hermitian $q \times q$ matrix-valued measure which is defined on $\mathscr{B}$ the Borel family of subsets of the m-dimensional torus T_m. We will also denote this measure by $\mathbf{F}$. The proof of the following theorem is immediate from the results given in [8, Section 3].

2.1. Theorem. *Let T_m, $\mathscr{B}$, $\mathbf{F}$ be defined as before. Then $\mathbf{L}_2(T_m, \mathscr{B}, \mathbf{F})$ the set of all $p \times q$ matrix-valued $\mathbf{\Phi}$ for which the integral*

$$\int_{T_m} \mathbf{\Phi}\, d\mathbf{F}\mathbf{\Phi}^* \qquad (* = \text{conjugation}),$$

exists is a Hilbert space under the inner product

$$((\mathbf{\Phi}, \mathbf{\Psi}))_{\mathbf{F}} = \operatorname{tr}(\mathbf{\Phi}, \mathbf{\Psi})_{\mathbf{F}}, \qquad (\mathbf{\Phi}, \mathbf{\Psi})_{\mathbf{F}} = \left(\frac{1}{2\pi}\right)^m \int_{T_m} \mathbf{\Phi}\, d\mathbf{F}\mathbf{\Psi}^*;$$

(b) *the mapping $\mathbf{V} : e^{-i(k,x)}\mathbf{I} \to \mathbf{X}_k$ $\{\mathbf{I}$ is the identity matrix$\}$ is an isometry between $\mathbf{L}_2(T_m, \mathscr{B}, \mathbf{F})$ and the closed subspace $\mathscr{S}\{\mathbf{X}_k, k \in Z^m\} \subseteq \mathscr{H}^q$, and this isometry is onto.*

For any matrix $\mathbf{G}$, we write $\mathbf{G}^-$ for the generalized inverse of $\mathbf{G}$ {cf. [9]}. If μ is a σ-finite nonnegative real-valued measure on $\mathscr{B}$ with respect to (w.r.t.) which $\mathbf{F}$ is absolutely continuous (a.c.), then it easily follows that $(d\mathbf{F}/d\mu)^-$ is a $\mathscr{B}$-measurable matrix-valued function.

The following lemma was proved in [10] and is stated here for completeness.

2.2. Lemma. *Let* (i) *$\mathbf{M}$ and $\mathbf{N}$ be $p \times q$ matrix-valued measures on $\mathscr{B}$.* (ii) *μ and ν be σ-finite nonnegative real-valued measures on $\mathscr{B}$* w.r.t. *which $\mathbf{M}$, $\mathbf{N}$ and $\mathbf{F}$ are* a.c. *Then*

(a) $$\int_{T_m} \frac{d\mathbf{M}}{d\mu}\left(\frac{d\mathbf{F}}{d\mu}\right)^- \left(\frac{d\mathbf{N}}{d\mu}\right)^* d\mu \text{ exists}$$

iff

$$\int_{T_m} \frac{d\mathbf{M}}{d\nu}\left(\frac{d\mathbf{F}}{d\nu}\right)^- \left(\frac{d\mathbf{N}}{d\nu}\right)^* d\nu \text{ exists}$$

(b) *If these integrals exist, they are equal.*

Thus the following definition makes sense.

DEFINITION. Let $\mathbf{M}$, $\mathbf{N}$, $\mathbf{F}$ and μ be as in the previous lemma. Then

(a) We say that $(\mathbf{M}, \mathbf{N})$ is Hellinger integrable w.r.t. $\mathbf{F}$ iff

$$\int_{T_m} \frac{d\mathbf{M}}{d\mu} \left(\frac{d\mathbf{F}}{d\mu}\right)^{-} \left(\frac{d\mathbf{N}}{d\mu}\right)^{*} d\mu \text{ exists.}$$

We write

$$\int_{T_m} \frac{d\mathbf{M}\, d\mathbf{N}^*}{d\mathbf{F}} = \int_{T_m} \frac{d\mathbf{M}}{d\mu} \left(\frac{d\mathbf{F}}{d\mu}\right)^{-} \left(\frac{d\mathbf{N}}{d\mu}\right)^{*} d\mu,$$

$$(\mathbf{M}, \mathbf{N})_{\mathbf{F}} = \left(\frac{1}{2\pi}\right)^{m} \int_{T_m} \frac{d\mathbf{M}\, d\mathbf{N}^*}{d\mathbf{F}};$$

(b) $\mathbf{H}_2(T_m, \mathscr{B}, \mathbf{F})$ is the class of all $p \times q$ matrix-valued measures $\mathbf{M}$ on $\mathscr{B}$ for which $\int_{T_m} (d\mathbf{M}\, d\mathbf{M}^*)/d\mathbf{F}$ exist.

The following theorem is proved in [10].

2.4. THEOREM. *Let $\mathbf{H}_2(T_m, \mathscr{B}, \mathbf{F})$ be as before. Then*

(a) $\mathbf{M}, \mathbf{F} \in \mathbf{H}_2(T_m, \mathscr{B}, \mathbf{F})$ *and* $\mathbf{A}, \mathbf{B}$ *matrices* $\Rightarrow \mathbf{AM} + \mathbf{BN} \in \mathbf{H}_2(T_m, \mathscr{B}, \mathbf{F})$.

(b) $M, \mathbf{N} \in \mathbf{H}_2(T_2, \mathscr{B}, \mathbf{F}) \Rightarrow (\mathbf{M}, \mathbf{N})$ *is Hellinger-integrable* w.r.t. $\mathbf{F}$.

(c) $\mathbf{H}_2(T_m, \mathscr{B}, \mathbf{F})$ *is a Hilbert space under the inner product* $\operatorname{tr}(\mathbf{M}, \mathbf{N})_{\mathbf{F}}$.

3. INTERPOLATION OF HOMOGENEOUS RANDOM FIELDS

Let $\{\mathbf{X}_k, k \in Z^m\}$ be a q-variate homogeneous random field over Z^m. T_m, $\mathscr{B}$ and $\mathbf{F}$ will be as before. Let K be any bounded subset of Z^m. K' will denote the complement of K in Z^m. $\mathscr{M}_K$ and $\mathscr{M}_{K'}$ will denote the (closed) subspaces spanned by $\mathbf{X}_k$, $k \in K$ and $\mathbf{X}_k$, $k \in K'$, respectively, i.e., $\mathscr{M}_K = \mathscr{S}\{\mathbf{X}_k, k \in K\}$ and $\mathscr{M}_{K'} = \mathscr{S}\{\mathbf{X}_k, k \in K'\}$. $\mathscr{M}_\infty$ will denote $\mathscr{S}\{\mathbf{X}_k, k \in Z^m\}$ and finally $\mathscr{N}_K$ will denote $\mathscr{M}_\infty \cap \mathscr{M}_{K'}^{\perp}$, where $\mathscr{M}_{K'}^{\perp}$ denote the orthogonal complement of $\mathscr{M}_{K'}$ in the fixed Hilbert space $\mathscr{H}^q$ containing the homogeneous random field $\mathbf{X}_k$.

3.1. DEFINITION. We say that

(a) K is interpolable w.r.t. $\{\mathbf{X}_k, k \in Z^m\}$ if $\mathscr{N}_K = \{\mathbf{0}\}$.

(b) $\{\mathbf{X}_k, k \in Z^m\}$ is interpolable if each bounded subset K of Z^m is interpolable w.r.t. $\{\mathbf{X}_k, k \in Z^m\}$.

(c) $\{\mathbf{X}_k, k \in Z^m\}$ is minimal of for each $k \in Z^m$, $\{k\}$ is not interpolable w.r.t. $\{\mathbf{X}_k, k \in Z^m\}$.

If $\mathbf{X} \in \mathcal{N}_K$, then $(\mathbf{X}, \mathbf{X}_k) = 0$ for all $k \in K'$. Thus the following definition makes sense.

3.2. DEFINITION. (a) For each $\mathbf{X} \in \mathcal{N}_K$ and each $e^{i\theta} \in T_m$, we define $\mathbf{P_X}$ on T_m by

$$\mathbf{P_X}(e^{i\theta}) = \sum (\mathbf{X}, \mathbf{X}_k)\, e^{-i(k,\theta)}.$$

(b) We define the operator $\mathbf{T}_K$ on $\mathcal{N}_K$ into $\mathbf{H}_2(T_m, \mathscr{B}, \mathbf{F})$ as follows: for each $\mathbf{X} \in \mathcal{N}_K$

$$\mathbf{T}_K \mathbf{X} = \mathbf{M}_{\mathbf{P_X}},$$

where for any trig-polynomial $\mathbf{P}$ with matrix coefficients the measure $\mathbf{M_P}$ on $\mathscr{B}$ is given by $\mathbf{M_P}(B) = \int_B \mathbf{P}(e^{i\theta})\, d\theta$.

The operator $\mathbf{T}_K$ has the following interesting properties.

3.3. THEOREM. (a) *Let* $\mathbf{X} \in \mathcal{N}_K$ *and* $\mathbf{\Psi}$ *be in* $\mathbf{L}_2(T_m, \mathscr{B}, \mathbf{F})$ such that $\mathbf{V\Psi} = \mathbf{X}$, *where* $\mathbf{V}$ *is the isomorphism on* $\mathbf{L}_2(T_m, \mathscr{B}, \mathbf{F})$ *onto* $\mathcal{M}_\infty$ {cf. Theorem 2.1(b)}. *Then for each* $B \in \mathscr{B}$, $\mathbf{M}_{\mathbf{P_X}}(B) = \int_B \mathbf{\Psi}\, d\mathbf{F}$.

(b) $\mathbf{T}_K$ *is an isometry on* $\mathcal{N}_K$ *into* $\mathbf{H}_2(T_m, \mathscr{B}, \mathbf{F})$.

In fact for all $\mathbf{X}$ *and* $\mathbf{Y}$ *in* $\mathcal{N}_K$

$$(\mathbf{X}, \mathbf{Y}) = (\mathbf{T}_K\mathbf{X}, \mathbf{T}_K\mathbf{Y})_{\mathbf{F}}.$$

(c) *The range of* $\mathbf{T}_K$ *is a closed subspace of the Hilbert space* $\mathbf{H}_2(T_m, \mathscr{B}, \mathbf{F})$.

PROOF. (a) Let $\mathbf{\Psi} \in \mathbf{L}_2(T_m, \mathscr{B}, \mathbf{F})$ and $\mathbf{X} = \mathbf{V\Psi}$. Then {cf. [8, p. 297]}

$$(\mathbf{X}, \mathbf{X}_k) = (\mathbf{\Psi}, e^{-i(k,\theta)}\mathbf{I})_{\mathbf{F}} = \left(\frac{1}{2\pi}\right)^m \int_{T_m} \mathbf{\Psi}\, d\mathbf{F} e^{i(k,\theta)}\, d\theta. \tag{1}$$

Also by definition of $\mathbf{M}_{\mathbf{P_X}}$,

$$\begin{aligned}
\left(\frac{1}{2\pi}\right)^m \int_{T_m} e^{i(k,\theta)}\, d\mathbf{M}_{\mathbf{P_X}}(e^{i\theta}) &= \left(\frac{1}{2\pi}\right)^m \int_{T_m} \mathbf{P_X}(e^{i\theta})\, e^{i(k,\theta)}\, d\theta \\
&= \left(\frac{1}{2\pi}\right)^m \int_{T_m} \left\{\sum_{n\in K} (\mathbf{X}, \mathbf{X}_n)\, e^{-i(n,\theta)}\right\} e^{i(k,\theta)}\, d\theta \\
&= \sum_{n\in K} \left(\frac{1}{2\pi}\right)^m \int_{T_m} (\mathbf{X}, \mathbf{X}_n)\, e^{i(k-n,\theta)}\, d\theta \\
&= (\mathbf{X}, \mathbf{X}_k).
\end{aligned} \tag{2}$$

By (1) and (2), the measures $\int_B \boldsymbol{\Psi}\, d\mathbf{F}$ and $\int_B \mathbf{P_X}(e^{i\theta})\, d\theta$ have the same Fourier coefficients and hence for each $B \in \mathscr{B}$

$$\mathbf{M}_{\mathbf{P_X}}(B) = \int_B \boldsymbol{\Psi}\, d\mathbf{F}.$$

(b) Let $\mathbf{X}$ and $\mathbf{Y}$ be in $\mathscr{N}_K$, and let $\boldsymbol{\Phi}$ and $\boldsymbol{\Psi}$ be in $\mathbf{L}_2(T_m, \mathscr{B}, \mathbf{F})$ such that $\mathbf{V\Phi} = \mathbf{X}$ and $\mathbf{V\Phi} = \mathbf{Y}$. Then by [10, Theorem 1]

$$(\mathbf{T}_K\mathbf{X}, \mathbf{T}_K\mathbf{Y})_\mathbf{F} = (\boldsymbol{\Phi}, \boldsymbol{\Psi})_\mathbf{F}. \tag{3}$$

Also {cf. [8, p. 297]}

$$(\mathbf{X}, \mathbf{Y}) = (\boldsymbol{\Phi}, \boldsymbol{\Psi})_\mathbf{F}. \tag{4}$$

From (3) and (4), (b) follows.

(c) Since $\mathscr{N}_K$ is a closed subspace and since by (b) $\mathbf{T}_K$ is an isometry on $\mathscr{N}_K$ into $\mathbf{H}_2(T_m, \mathscr{B}, \mathbf{F})$, therefore the range of $\mathbf{T}_K$ is a closed subspace of $\mathbf{H}_2(T_m, \mathscr{B}, \mathbf{F})$. (Q.E.D.)

3.4. Lemma. *Let* $\mathbf{X} \in \mathscr{N}_K \cap \mathscr{N}_L$. *Then*

$$\mathbf{T}_K\mathbf{X} = \mathbf{T}_L\mathbf{X}.$$

Proof. It is clear that $\mathscr{N}_K \cap \mathscr{N}_L = \mathscr{N}_{K\cup L}$. Hence $\mathbf{T}_K\mathbf{X} = \mathbf{T}_{K\cap L}\mathbf{X} = \mathbf{T}_L\mathbf{X}$. (Q.E.D.)

Making use of this lemma, $\mathbf{T}_K$'s may be sewed to introduce a well-defined operators $\mathbf{T}$. This done in the following theorem.

3.5. Theorem. *Let* $\mathscr{N} = \cup\, \mathscr{N}_K$, *where* K *is a bounded set in* Z^m. *Define the operator* $\mathbf{T}$ *on* $\mathscr{N}$ *by*

$$\mathbf{TX} = \mathbf{T}_K\mathbf{X} \qquad \textit{if} \qquad \mathbf{X} \in \mathscr{N}_K.$$

Then

(a) $\mathscr{N}$ *is a linear manifold in* $\mathscr{M}_\infty$, *i.e.*,

$$\mathbf{X}, \mathbf{Y} \in \mathscr{N} \qquad \textit{and} \qquad \mathbf{A}, \mathbf{B} \textit{ matrices} \Rightarrow \mathbf{AX} + \mathbf{BY} \in \mathscr{N}.$$

(b) $\mathbf{T}$ *is a single-valued linear operator on* $\mathscr{N}$, *i.e., if* $\mathbf{X}, \mathbf{Y} \in \mathscr{N}$ *and* $\mathbf{A}, \mathbf{B}$ *are matrices, then*

$$\mathbf{T}(\mathbf{AX} + \mathbf{BY}) = \mathbf{ATX} + \mathbf{BTY}.$$

(c) $\mathbf{T}$ *is an isometry on* $\mathscr{N}$ *into* $\mathbf{H}_2(T_m, \mathscr{B}, \mathbf{F})$. *In fact for* $\mathbf{X}, \mathbf{Y} \in \mathscr{N}$,

$$(\mathbf{X}, \mathbf{Y}) = (\mathbf{TX}, \mathbf{TY})_\mathbf{F}.$$

(d) *The range of* $\mathbf{T}$ *consists of all matrix-valued measures* $\mathbf{M_P}$ *for which the Hellinger integrals* $\int_{T_m} (d\mathbf{M_P}\, d\mathbf{M_P^*})/d\mathbf{F}$ *exist, where* $\mathbf{P}$ *is a polynomial with matrix coefficients and* $\mathbf{M_P}$ *is related to* $\mathbf{P}$ *as in Definition* 3.2(b).

PROOF. (a) follows from the fact that $\mathcal{N}_K \cup \mathcal{N}_L \subseteq \mathcal{N}_{K\cup L}$. (b) and (c) are consequences of Theorem 3.3(b) and Lemma 3.4. (d) Let $\mathbf{X} \in \mathcal{N}$. Then $\mathbf{X} \in \mathcal{N}_K$ for some K. It then follows from the definition of $\mathbf{T}$ that

$$\mathbf{TX} = \mathbf{T}_K\mathbf{X} = \mathbf{M}_{\mathbf{P}_\mathbf{X}}.$$

By (1) and (c) it follows that $(\mathbf{X}, \mathbf{X}) = (\mathbf{M}_{\mathbf{P}_\mathbf{X}}, \mathbf{M}_{\mathbf{P}_\mathbf{X}})_\mathbf{F}$ and hence $(\mathbf{M}_{\mathbf{P}_\mathbf{X}}, \mathbf{M}_{\mathbf{P}_\mathbf{X}})$ is Hellinger integrable w.r.t. $\mathbf{F}$.

Conversely let $\mathbf{M}_P$ be a matrix-valued measure such that $\int_{T_m} (d\mathbf{M}_\mathbf{P}\, d\mathbf{M}_\mathbf{P}^*)/d\mathbf{F}$ exists, where for each $B \in \mathscr{B}$

$$\mathbf{M}_\mathbf{P}(B) = \int_B \mathbf{P}(e^{i\theta})\, d\theta.$$

Then {cf. [10, Theorem 1]} $\mathbf{\Phi} = (d\mathbf{M}_\mathbf{P}/d\mu)\,(d\mathbf{F}/d\mu)^- \in \mathbf{L}_2(T_m, \mathscr{B}, \mathbf{F})$, where μ is any σ-finite nonnegative real-valued measure w.r.t. which $\mathbf{M}_\mathbf{P}$ and $\mathbf{F}$ are a.c. If $\mathbf{X} \in \mathcal{M}_\infty$ such that $\mathbf{V\Phi} = \mathbf{X}$, where $\mathbf{V}$ is as in Theorem 3.3, then {cf. [8, p. 297] and [10, Theorem 1]}

$$\begin{aligned}
(\mathbf{X}, \mathbf{X}_k) &= (\mathbf{\Phi}, e^{-i(k,\theta)})_\mathbf{F} \\
&= \left(\frac{1}{2\pi}\right)^m \int_{T_m} \frac{d\mathbf{M}_\mathbf{P}}{d\mu}\left(\frac{d\mathbf{F}}{d\mu}\right)^- \frac{d\mathbf{F}}{d\mu}\, e^{i(k,\theta)}\, d\mu \\
&= \left(\frac{1}{2\pi}\right)^m \int_{T_m} e^{i(k,\theta)} \frac{d\mathbf{M}_\mathbf{P}}{d\mu}\, d\mu \\
&= \left(\frac{1}{2\pi}\right)^m \int_{T_m} e^{i(k,\theta)}\, d\mathbf{M}_\mathbf{P} \\
&= \left(\frac{1}{2\pi}\right)^m \int_{T_m} e^{i(k,\theta)}\mathbf{P}(e^{i\theta})\, d\theta
\end{aligned}$$

But

$$\mathbf{P}_\mathbf{X}(e^{i\theta}) = \sum (\mathbf{X}, \mathbf{X}_k)\, e^{-i(k,\theta)}. \tag{3}$$

By (2) and (3) we have that

$$\mathbf{P}(e^{i\theta}) = \mathbf{P}_\mathbf{X}(e^{i\theta}).$$

It is clear that $\mathbf{M}_\mathbf{P} = \mathbf{M}_{\mathbf{P}_\mathbf{X}}$ and the result follows. (Q.E.D.)

We are now ready to give a characterization for the interpolability of a homogeneous random field.

3.6. THEOREM. *The q-variate homogeneous random field $\{\mathbf{X}_k, k \in Z^m\}$ is interpolable iff for any matrix-valued function $\mathbf{P}$ for which $\mathbf{M}_\mathbf{P}$ is not a null point in $\mathbf{H}_2(T_m, \mathscr{B}, \mathbf{F})$, $(\mathbf{M}_\mathbf{P}, \mathbf{M}_\mathbf{P})$ is not Hellinger-integrable* w.r.t. $\mathbf{F}$.

PROOF. ($\Leftarrow$) If K is any bounded subset of Z^m, it is a consequence of Theorem 3.5(d) that $\mathcal{N}_K = \{0\}$. Hence by Definition 3.1(a) K is interpolable w.r.t. $\{\mathbf{X}_k, k \in Z^m\}$. Since K is arbitrary it follows that $\mathcal{N} = \bigcup_K \mathcal{N}_K = \{0\}$ so that by Definition 3.1(b), $\{\mathbf{X}_k, k \in Z^m\}$ is interpolable. ($\Rightarrow$) It follows that $\mathcal{N} = \{0\}$. Hence by Theorem 3.5(d) range of $\mathbf{T} = \{0\}$. The result follows from Theorem 3.5(c). (Q.E.D.)

The following theorem is a generalization of results concerning q-variate stationary stochastic processes {cf. [Theorem 2]}.

3.7. THEOREM. *Let $\mathbf{Z}_k$ be the orthogonal projection of $\mathbf{X}_k$ onto the subspace $\mathcal{S}^{\perp}\{\mathbf{X}_n, n \neq k\}$, and let $\mathbf{Y}_k = (\mathbf{X}_0, \mathbf{X}_0)^- \mathbf{Z}_k$, where $(\mathbf{Z}_0, \mathbf{Z}_0)^-$ is the generalized inverse of $(\mathbf{Z}_0, \mathbf{Z}_0)$. Then*

(a) $$(\mathbf{Z}_0, \mathbf{Z}_0)^- = (\mathbf{Y}_0, \mathbf{Y}_0) = \left(\frac{1}{2\pi}\right)^m \int_{T_m} \frac{d\mathbf{M}_\mathbf{J}\, d\mathbf{M}_\mathbf{J}}{d\mathbf{F}}$$

$$(\mathbf{Z}_0, \mathbf{Z}_0) = \left[\left(\frac{1}{2\pi}\right)^m \int_{T_m} \frac{d\mathbf{M}_\mathbf{J}\, d\mathbf{M}_\mathbf{J}}{d\mathbf{F}}\right]^-,$$

where $\mathbf{J}$ is the projection matrix on the subspace $\mathscr{C}^q$ of q-tupes of complex numbers onto the range of $(\mathbf{Z}_0, \mathbf{Z}_0)$ in the priviledge basis of $\mathscr{C}^q$.

(b) $$\{\mathbf{X}_k, k \in Z^m\} \quad \text{is minimal iff} \quad \int_{T_m} \frac{d\mathbf{M}_\mathbf{J}\, d\mathbf{M}_\mathbf{J}}{d\mathbf{F}} \neq \mathbf{0}.$$

(c) *$\{\mathbf{Y}_k, k \in Z^m\}$ is a q-variate homogeneous random field over Z^m with the spectral distribution function*

$$\left(\frac{1}{2\pi}\right)^m \int_0^\theta \frac{d\mathbf{M}_\mathbf{J}\, d\mathbf{M}_\mathbf{J}}{d\mathbf{F}}.$$

(d) *$\{\mathbf{Y}_k, k \in Z^m\}$ and $\{\mathbf{X}_k, k \in Z^m\}$ are biorthogonal, i.e.,*

$$(\mathbf{Y}_m, \mathbf{X}_n) = \delta_{mn}\mathbf{J}.$$

PROOF. (a) By Theorem 2.5, $(\mathbf{Z}_0, \mathbf{Z}_0) = (\mathbf{M}_{\mathbf{Z}_0}, \mathbf{M}_{\mathbf{Z}_0})_\mathbf{F}$, where for each $B \in \mathscr{B}$, $\mathbf{M}_{\mathbf{Z}_0}(B) = \int_B (\mathbf{Z}_0, \mathbf{Z}_0)\, d\theta$. Hence

$$\begin{aligned}(\mathbf{Z}_0, \mathbf{Z}_0)^- &= (\mathbf{Z}_0, \mathbf{Z}_0)^- (\mathbf{Z}_0, \mathbf{Z}_0) (\mathbf{Z}_0, \mathbf{Z}_0)^- \\ &= (\mathbf{Z}_0, \mathbf{Z}_0)^- (\mathbf{M}_{\mathbf{Z}_0}, \mathbf{M}_{\mathbf{Z}_0})_\mathbf{F} (\mathbf{Z}_0, \mathbf{Z}_0)^- \\ &= \left(\frac{1}{2\pi}\right)^m \int_{T_m} \frac{d\mathbf{M}_\mathbf{J}\, d\mathbf{M}_\mathbf{J}}{d\mathbf{F}}.\end{aligned}$$

Consequently

$$(\mathbf{Z}_0, \mathbf{Z}_0)^- = (\mathbf{Y}_0, \mathbf{Y}_0) = \left(\frac{1}{2\pi}\right)^m \int_{T_m} \frac{d\mathbf{M}_\mathbf{J}\, d\mathbf{M}_\mathbf{J}}{d\mathbf{F}}$$

and

$$(\mathbf{Z}_0, \mathbf{Z}_0) = \left[\left(\frac{1}{2\pi}\right)^m \int_{T_m} \frac{d\mathbf{M}_\mathbf{J}\, d\mathbf{M}_\mathbf{J}}{d\mathbf{F}}\right]^-.$$

(b) By (a),

$$(\mathbf{Z}_0, \mathbf{Z}_0) = \left[\left(\frac{1}{2\pi}\right)^m \int_{T_m} \frac{d\mathbf{M}_\mathbf{J}\, d\mathbf{M}_\mathbf{J}}{d\mathbf{F}}\right]^-.$$

From this and Definition 3.1 (c) we have (b).

(c) Obviously $\{\mathbf{Y}_k, k \in Z^m\}$ is a q-variate homogeneous random field. Hence by (a) $(\mathbf{Y}_0, \mathbf{Y}_0) = (\mathbf{Z}_0, \mathbf{Z}_0)^- = (\mathbf{M}_\mathbf{J}, \mathbf{M}_\mathbf{J})_\mathbf{F}$. It follows that the spectral distribution of $(\mathbf{Y}_k, k \in Z^m)$ is

$$\left(\frac{1}{2\pi}\right)^m \int_0^\theta \frac{d\mathbf{M}_\mathbf{J}\, d\mathbf{M}_\mathbf{J}}{d\mathbf{F}}.$$

(d)
$$\begin{aligned}(\mathbf{Y}_0, \mathbf{X}_0) &= ((\mathbf{Z}_0, \mathbf{Z}_0)^- \mathbf{Z}_0, \mathbf{X}_0)\\ &= (\mathbf{Z}_0, \mathbf{Z}_0)^- (\mathbf{Z}_0, \mathbf{X}_0)\\ &= (\mathbf{Z}_0, \mathbf{Z}_0)^- (\mathbf{Z}_0, \mathbf{Z}_0) = \mathbf{J}.\end{aligned}$$

For $m \neq 0$, $\mathbf{Z}_m \perp \mathscr{S}\{\mathbf{X}_n, n \neq m\}$, therefore $(\mathbf{Y}_m, \mathbf{X}_0) = 0$. Hence $(\mathbf{Y}_m, \mathbf{X}_n) = \delta_{mn}\mathbf{J}$. (Q.E.D.)

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